$$\text{Solve} \qquad \left(\frac{U_k + \frac{i}{2}}{U_k - \frac{i}{2}}\right)^L = \prod_{\substack{j=1 \\ j \neq k}}^{M} \frac{U_k - U_j + i}{U_k - U_j - i}$$

Then

$$E = 2 \sum_{k=1}^{M} \frac{1}{U_k^2 + \frac{1}{4}} \qquad V = \prod_{i=1}^{M} \frac{U_k + \frac{i}{2}}{U_k - \frac{i}{2}}$$

```
Bethe[L_, M_] :=

  unknowns = Table[u[k], {k, M}];
  eqns = Table[

      (u[k] + I / 2)^L       M   u[k] - u[j] + I
      ---------------    == -∏  -----------------,
      (u[k] - I / 2)        j=1  u[k] - u[j] - I

    {k, M}
  ];
  sols = Select[
    Solve[eqns, unknowns],
    Length[Union[Last /@ #]] == M &
  ];
         M        1
  Es = 2 ∑  --------------   /. sols;
        k=1  u[k]^2 + 1 / 4

  {eqns, sols, Es // N}
```

Bethe[4, 0]

{{}, {{}}, {0.}}

Bethe[4, 1]

$$\left\{\left\{\frac{\left(\frac{i}{2} + u[1]\right)^4}{\left(-\frac{i}{2} + u[1]\right)^4} == 1\right\}, \left\{\left\{u[1] \to -\frac{1}{2}\right\}, \{u[1] \to 0\}, \left\{u[1] \to \frac{1}{2}\right\}\right\}, \{4., 8., 4.\}\right\}$$

Bethe[4, 2]

$$\left\{\left\{\frac{\left(\frac{i}{2} + u[1]\right)^4}{\left(-\frac{i}{2} + u[1]\right)^4} == \frac{i + u[1] - u[2]}{-i + u[1] - u[2]}, \frac{\left(\frac{i}{2} + u[2]\right)^4}{\left(-\frac{i}{2} + u[2]\right)^4} == \frac{i - u[1] + u[2]}{-i - u[1] + u[2]}\right\},\right.$$

$$\left.\left\{\left\{u[1] \to -\frac{1}{2\sqrt{3}}, u[2] \to \frac{1}{2\sqrt{3}}\right\}, \left\{u[1] \to \frac{1}{2\sqrt{3}}, u[2] \to -\frac{1}{2\sqrt{3}}\right\}\right\}, \{12., 12.\}\right\}$$

Bethe[4, 3]

Solve::svars : Equations may not give solutions for all "solve" variables. ≫

$$\left\{\left\{\frac{\left(\frac{i}{2}+u[1]\right)^4}{\left(-\frac{i}{2}+u[1]\right)^4} == \frac{(i+u[1]-u[2])\ (i+u[1]-u[3])}{(-i+u[1]-u[2])\ (-i+u[1]-u[3])}\,,\right.\right.$$

$$\frac{\left(\frac{i}{2}+u[2]\right)^4}{\left(-\frac{i}{2}+u[2]\right)^4} == \frac{(i-u[1]+u[2])\ (i+u[2]-u[3])}{(-i-u[1]+u[2])\ (-i+u[2]-u[3])}\,,$$

$$\left.\frac{\left(\frac{i}{2}+u[3]\right)^4}{\left(-\frac{i}{2}+u[3]\right)^4} == \frac{(i-u[1]+u[3])\ (i-u[2]+u[3])}{(-i-u[1]+u[3])\ (-i-u[2]+u[3])}\right\}\,,$$

$$\left\{\left\{u[1]\to-\frac{1}{2}\,,\ u[2]\to\frac{1}{2}\left(-1-\sqrt{2}\right)\,,\ u[3]\to-\frac{1}{2}+\frac{1}{\sqrt{2}}\right\}\,,\right.$$

$$\left\{u[1]\to-\frac{1}{2}\,,\ u[2]\to\frac{1}{2}\left(-1-i\sqrt{2}\right)\,,\ u[3]\to\frac{1}{2}\left(-1+i\sqrt{2}\right)\right\}\,,$$

$$\left\{u[1]\to-\frac{1}{2}\,,\ u[2]\to\frac{1}{2}\left(-1+i\sqrt{2}\right)\,,\ u[3]\to\frac{1}{2}\left(-1-i\sqrt{2}\right)\right\}\,,$$

$$\left\{u[1]\to-\frac{1}{2}\,,\ u[2]\to\frac{1}{2}\left(-1+\sqrt{2}\right)\,,\ u[3]\to\frac{1}{2}\left(-1-\sqrt{2}\right)\right\}\,,$$

$$\left\{u[1]\to0\,,\ u[2]\to-\frac{i\sqrt{5}}{2}\,,\ u[3]\to\frac{i\sqrt{5}}{2}\right\}\,,\ \left\{u[1]\to0\,,\ u[2]\to\frac{i\sqrt{5}}{2}\,,\ u[3]\to-\frac{i\sqrt{5}}{2}\right\}\,,$$

$$\left\{u[1]\to\frac{1}{2}\,,\ u[2]\to\frac{1}{2}\left(1-\sqrt{2}\right)\,,\ u[3]\to\frac{1}{2}+\frac{1}{\sqrt{2}}\right\}\,,$$

$$\left\{u[1]\to\frac{1}{2}\,,\ u[2]\to\frac{1}{2}\left(1-i\sqrt{2}\right)\,,\ u[3]\to\frac{1}{2}\left(1+i\sqrt{2}\right)\right\}\,,$$

$$\left\{u[1]\to\frac{1}{2}\,,\ u[2]\to\frac{1}{2}\left(1+i\sqrt{2}\right)\,,\ u[3]\to\frac{1}{2}\left(1-i\sqrt{2}\right)\right\}\,,$$

$$\left\{u[1]\to\frac{1}{2}\,,\ u[2]\to\frac{1}{2}\left(1+\sqrt{2}\right)\,,\ u[3]\to\frac{1}{2}\left(1-\sqrt{2}\right)\right\}\,,$$

$$\left\{u[1]\to-\frac{1}{2\sqrt{3}}\,,\ u[2]\to\frac{1}{4}\left(-\sqrt{3}-i\sqrt{13}\right)\,,\ u[3]\to\frac{1}{4}\left(-\sqrt{3}+i\sqrt{13}\right)\right\}\,,$$

$$\left\{u[1]\to-\frac{1}{2\sqrt{3}}\,,\ u[2]\to\frac{1}{4}\left(-\sqrt{3}+i\sqrt{13}\right)\,,\ u[3]\to\frac{1}{4}\left(-\sqrt{3}-i\sqrt{13}\right)\right\}\,,$$

$$\left\{u[1]\to\frac{1}{2\sqrt{3}}\,,\ u[2]\to\frac{1}{4}\left(\sqrt{3}-i\sqrt{13}\right)\,,\ u[3]\to\frac{1}{4}\left(\sqrt{3}+i\sqrt{13}\right)\right\}\,,$$

$$\left.\left\{u[1]\to\frac{1}{2\sqrt{3}}\,,\ u[2]\to\frac{1}{4}\left(\sqrt{3}+i\sqrt{13}\right)\,,\ u[3]\to\frac{1}{4}\left(\sqrt{3}-i\sqrt{13}\right)\right\}\right\}\,,$$

$$\{12.,\ 4.+0.\ i,\ 4.+0.\ i,\ 12.,\ 4.,\ 12.,\ 4.+0.\ i,\ 4.+0.\ i,$$
$$\left.12.,\ 4.+0.\ i,\ 4.+0.\ i,\ 4.+0.\ i,\ 4.+0.\ i\}\right\}$$

Bethe[5, 0]

$\{\{\},\ \{\{\}\},\ \{0.\}\}$

Bethe[5, 1]

$$\left\{\left\{\frac{\left(\frac{i}{2} + u[1]\right)^5}{\left(-\frac{i}{2} + u[1]\right)^5} == 1\right\},\right.$$

$$\left\{\left\{u[1] \to -\sqrt{\frac{1}{4} + \frac{1}{2\sqrt{5}}}\right\}, \left\{u[1] \to \sqrt{\frac{1}{4} + \frac{1}{2\sqrt{5}}}\right\}, \left\{u[1] \to -\frac{1}{2}\sqrt{\frac{1}{5}\left(5 - 2\sqrt{5}\right)}\right\},\right.$$

$$\left\{u[1] \to \frac{1}{2}\sqrt{\frac{1}{5}\left(5 - 2\sqrt{5}\right)}\right\}\right\}, \{2.76393, 2.76393, 7.23607, 7.23607\}\right\}$$